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**M-theory on manifolds of G_2 holonomy:
the first twenty years¹**

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Abstract

In 1981, covariantly constant spinors were introduced into Kaluza-Klein theory as a way of counting the number of supersymmetries surviving compactification. These are related to the *holonomy* group of the compactifying manifold. The first non-trivial example was provided in 1982 by $D = 11$ supergravity on the squashed S^7 , whose G_2 holonomy yields $N = 1$ in $D = 4$. In 1983 another example was provided by $D = 11$ supergravity on $K3$, whose $SU(2)$ holonomy yields half the maximum supersymmetry. In 2002, G_2 and $K3$ manifolds continue to feature prominently in the full $D = 11$ M-theory and its dualities. In particular, singular G_2 compactifications can yield chiral ($N = 1, D = 4$) models with realistic gauge groups. The notion of generalized holonomy is also discussed.

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1 Introduction

For the *Supergravity@25* conference, the organizers requested that the speakers provide a blend of something historical and something topical. So I have chosen to speak about *holonomy*, especially G_2 holonomy.

In 1981, Witten laid down the criterion for spacetime supersymmetry in Kaluza-Klein theory [1]. The number of spacetime supersymmetries is given by the number of covariantly constant spinors on the compactifying manifold. Covariantly constant spinors are, in their turn, related to the *holonomy*^a group of the corresponding connection. It was well known that the number of massless gauge bosons was determined by the isometry group of the compactifying manifold, but it turned out to be the holonomy group that determined the number of massless gravitinos. This was further emphasized in [2].

The first non-trivial example was provided in 1982 [2] by compactifying $D = 11$ supergravity on the squashed S^7 [5], an Einstein space whose G_2 holonomy yields $N = 1$ in $D = 4$. (The round S^7 has trivial holonomy and hence yields the maximum $N = 8$ supersymmetry [6].) Although the phenomenologically desirable $N = 1$ supersymmetry [1] and non-abelian gauge groups appear in four dimensions, the resulting theory was not realistic, being vectorlike with $SO(5) \times SO(3)$ and living on AdS_4 . It nevertheless provided valuable insight into the workings of modern Kaluza-Klein theories. Twenty years later, G_2 manifolds continue to play an important role in $D = 11$ M-theory for the same $N = 1$ reason. But the full M-theory, as opposed to its low energy limit of $D = 11$ supergravity, admits the possibility of *singular* G_2 compactifications which can yield *chiral* ($N = 1, D = 4$) models living in Minkowski space and with realistic gauge groups.

In 1983 another example was provided by $D = 11$ supergravity on $K3$ [7], whose $SU(2)$ holonomy yields half the maximum supersymmetry. For the first time, the Kaluza-Klein particle spectrum was dictated by the topology (Betti numbers and index theorems) rather than the geometry of the compactifying manifold, which was four-dimensional, Ricci flat and without isometries. It was thus a forerunner of the six-dimensional Ricci flat Calabi-Yau compactifications [8], whose $SU(3)$ holonomy yields ($N = 1, D = 4$) starting from ($N = 1, D = 10$). $K3$ compactifications also continue to feature prominently in M-theory and its dualities.

2 D=11 supergravity

The low energy limit of M-theory (or $D = 11$ supergravity as we used to call it) was introduced in 1978 by Cremmer, Julia and Scherk [9], not long after the discovery of supergravity itself [10, 11]. The unique $D = 11, N = 1$ supermultiplet is comprised of a graviton g_{MN} , a gravitino Ψ_M and 3-form gauge field A_{MNP} , where $M = 0, 1, \dots, 10$, with 44, 128 and 84 physical degrees of freedom, respectively.

^aHolonomy had already found its way into the physics literature via gravitational instantons [3] and non-linear sigma models [4].

Already the rank of the form, dictated by supersymmetry to be three, presages deep connections with 2-branes and indeed G_2 holonomy.

The supersymmetry tranformation rule of the gravitino reduces in a purely bosonic background to

$$\delta\Psi_M = \tilde{D}_M\epsilon \quad (1)$$

where the parameter ϵ is a 32-component anticommuting spinor, and where

$$\tilde{D}_M = D_M - \frac{i}{144} \left(\Gamma^{NPQR}{}_M + 8\Gamma^{PQR}\delta^N{}_M \right) F_{NPQR} \quad (2)$$

Here D_M is the usual Riemannian covariant derivative involving the usual Levi-Civita connection ω_M

$$D_M = \partial_M - \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB}, \quad (3)$$

Γ^A are the $D = 11$ Dirac matrices and $F = dA$.

The bosonic field equations are

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3} \left(F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F^{PQRS}F_{PQRS} \right) \quad (4)$$

and

$$d * F + F \wedge F = 0 \quad (5)$$

Being at Stonybrook, I should not forget the paper that I wrote with Peter van Nieuwenhuizen [12] pointing out that the 4-form field strength can thus give rise to a cosmological constant $\Lambda = -12m^2$ in the four dimensional subspace:

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma} \quad (6)$$

where $\mu = 0, 1, 2, 3$ and m is a constant with the dimensions of mass. A similar conclusion was reached independently by Aurilia et al [13]. This device was then used by Freund and Rubin [14] to effect a spontaneous compactification from $D = 11$ to $D = 4$, yielding the product of a four-dimensional spacetime with negative curvature

$$R_{\mu\nu} = -12m^2g_{\mu\nu} \quad (7)$$

and a seven-dimensional internal space of positive curvature

$$R_{mn} = 6m^2g_{mn} \quad (8)$$

where $m = 1, 2, \dots, 7$. Accordingly, the supercovariant derivative also splits as

$$\tilde{D}_\mu = D_\mu + m\gamma_\mu\gamma_5 \quad (9)$$

and

$$\tilde{D}_m = D_m - \frac{1}{2}m\Gamma_m \quad (10)$$

If we choose the spacetime to be maximally symmetric but leave the internal space X^7 arbitrary, we are led to the $D = 11$ geometry $AdS_4 \times X^7$. The first example

was provided by the choice $X^7 = \text{round } S^7$ [6, 15] which is maximally supersymmetric^b. The next example was the round S^7 with parallelizing torsion [17] which preserves no supersymmetry. However, it was also of interest to look for something in between, and this is where holonomy comes to the fore.

3 Killing spinors, holonomy and supersymmetry

The number of supersymmetries surviving compactification depends on the number of covariantly constant spinors [1]. To see this, we look for vacuum solutions of the field equations for which the the gravitino field Ψ vanishes. In order that the vacuum be supersymmetric, therefore, it is necessary that the gravitino remain zero when we perform a supersymmetry transformation and hence that the background supports spinors ϵ satisfying

$$\tilde{D}_M \epsilon = 0 \quad (11)$$

In the case of a product manifold, this reduces to

$$\tilde{D}_\mu \epsilon(x) = 0 \quad (12)$$

and

$$\tilde{D}_m \eta(y) = 0 \quad (13)$$

where $\epsilon(x)$ is a 4-component anticommuting spinor and $\eta(y)$ is an 8-component commuting spinor. The first equation is satisfied automatically with our choice of AdS_4 spacetime and hence the number of $D = 4$ supersymmetries, $0 \leq N \leq 8$, devolves upon the number of *Killing spinors* on X^7 . They satisfy the integrability condition

$$[\tilde{D}_m, \tilde{D}_n] \eta = -\frac{1}{4} C_{mn}{}^{ab} \Gamma_{ab} \eta = 0 \quad (14)$$

where $C_{mn}{}^{ab}$ is the Weyl tensor.

The subgroup of $Spin(7)$ (the double cover of the tangent space group $SO(7)$) generated by this linear combination of $Spin(7)$ generators Γ_{ab} corresponds to the *holonomy* group \mathcal{H} [2]. The number of supersymmetries, N , is then given by the number of singlets appearing in the decomposition of the 8 of $Spin(7)$ under \mathcal{H} [18]. Some examples are given in Table 1.

Strictly speaking, the Weyl tensor characterizes the *restricted* holonomy group of \tilde{D}_m . If the space is not simply connected there may be further global obstructions to the existence of unbroken supersymmetries. For example, solutions of the form T^7/Γ and S^7/Γ , where Γ is a discrete group, have vanishing Weyl tensor but admit fewer than 8 Killing spinors [7].

The phenomenological desirability of having just one Killing spinor, and hence just one four-dimensional supersymmetry, is also discussed in [1].

^bThe first Ricci flat ($m = 0$) example of a compactification of $D = 11$ supergravity was provided by the choice $X^7 = T^7$ [16] which is also maximally supersymmetric.

\mathcal{H}	$8 \rightarrow$	N
$Spin(7)$	8	0
G_2	$1 + 7$	1
$SU(3)$	$1 + 1 + 3 + \bar{3}$	2
$SU(2)$	$1 + 1 + 1 + 1 + 2 + \bar{2}$	4
$\mathbf{1}$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	8

Table 1: Examples of holonomy groups and the resulting supersymmetry

4 G_2

We see that the exceptional group G_2 is of particular interest since it yields just $N = 1$ supersymmetry. In fact, the first example of a Kaluza-Klein compactification with non-trivial holonomy was provided by the squashed S^7 which does indeed have $\mathcal{H} = G_2$ [2, 5].

This is probably a good time to say a word about terminology. When talking of the *holonomy* of a manifold, some authors take the word to refer exclusively to the Levi-Civita connection appearing in the Riemannian covariant derivative D_m . According to this definition, the holonomy of the squashed S^7 would be $Spin(7)$. The group G_2 would then correspond to what mathematicians call *weak holonomy* [19, 20]. A 7-dimensional Einstein manifold with $R_{mn} = 6m^2 g_{mn}$ has weak holonomy G_2 if it admits a 3-form A obeying

$$dA = 4m * A \quad (15)$$

That such a 3-form exists on the squashed S^7 can be proved by invoking the single (constant) Killing spinor η . The required 3-form is then given by [5]

$$A_{mnp} \sim \bar{\eta} \Gamma_{mnp} \eta \quad (16)$$

However, I prefer not to adopt this terminology for two reasons. First, from a strictly mathematical point of view, one should speak not of the holonomy of a manifold but rather of the connection on the manifold. Different connections on the same manifold can have different holonomies. Secondly, from a physical point of view, the whole reason for being interested in holonomy in the first place is because of supersymmetry, and here the relevant connection is not the Levi-Civita connection ω appearing in D_m but rather the generalized connection appearing in

$$\tilde{D}_m = \partial_m - \frac{1}{4} \omega_m^{ab} \Gamma_{ab} - \frac{1}{2} m e_m^a \Gamma_a \quad (17)$$

So in the context of M -theory, when I speak loosely of the holonomy of a manifold, it is the supersymmetric connection that I have in mind ^c.

^cSome early papers used the term *Weyl holonomy* which should probably now be abandoned.

Owing to this generalized connection, vacua with $m \neq 0$ present subtleties and novelties not present in the $m = 0$ case [25], for example the phenomenon of *skew-whiffing* [5, 15]. For each Freund-Rubin compactification, one may obtain another by reversing the orientation of X^7 . The two may be distinguished by the labels *left* and *right*. An equivalent way to obtain such vacua is to keep the orientation fixed but to make the replacement $m \rightarrow -m$. So the covariant derivative (17), and hence the condition for a Killing spinor, changes but the integrability condition (14) remains the same. With the exception of the round S^7 , where both orientations give $N = 8$, at most one orientation can have $N \geq 0$. This is the *skew-whiffing theorem*. A corollary is that other *symmetric spaces*, which necessarily admit an orientation-reversing isometry, can have no supersymmetries. Examples are provided by products of round spheres. Of course, skew-whiffing is not the only way to obtain vacua with less than maximal supersymmetry. A summary of known X^7 , their supersymmetries and stability properties is given in [15]. Note, however, that skew-whiffed vacua are automatically stable at the classical level since skew-whiffing affects only the spin $3/2$, $1/2$ and 0^- towers in the Kaluza-Klein spectrum, whereas the criterion for classical stability involves only the 0^+ tower [62, 15].

Once again the squashed S^7 provided the first non-trivial example: the left squashed S^7 has $N = 1$ but the right squashed S^7 has $N = 0$. Interestingly enough, this means that setting the suitably normalized 3-form (16) equal to the $D = 11$ supergravity 3-form provides a solution to the field equations, but only in the right squashed case. This solution is called the *right squashed S^7 with torsion* [5] since A_{mnp} may be interpreted as a Ricci-flattening torsion. Other examples were provided by the left squashed $N(1, 1)$ spaces [24], one of which has $N = 3$ and the other $N = 1$, while the right squashed counterparts both have $N = 0$.

All this presents a dilemma. If the Killing spinor condition changes but the integrability condition does not, how does one give a holonomic interpretation to the different supersymmetries? Indeed $N = 3$ is not allowed by the usual rules. The answer to this question may be found in a paper [21] written before we knew about skew-whiffing. The authors note that in (17), the $SO(7)$ generators Γ_{ab} , augmented by presence of Γ_a , together close on $SO(8)$. Hence one may introduce a generalized holonomy group $\mathcal{H}_{gen} \subset SO(8)$ and ask how the 8 of $SO(8)$ decomposes under \mathcal{H}_{gen} . In the case of the left squashed S^7 , $\mathcal{H}_{gen} = SO(7)^-$, $8 \rightarrow 1 + 7$ and $N = 1$, but for the right squashed S^7 , $\mathcal{H}_{gen} = SO(7)^+$, $8 \rightarrow 8$ and $N = 0$.

Kaluza-Klein compactification of supergravity was an active area of research in the early 1980s. Some early papers are [1, 2, 5, 6, 7, 17, 21, 22, 23, 26, 27] and I am glad to see many of the pioneers in the audience today: Leonardo Castellani, Bernard de Wit, Pietro Fre, Dan Freedman, Gary Gibbons, Bernard Julia, Ergin Sezgin, John Schwarz, Peter van Nieuwenhuizen, Nick Warner and Peter West. Reviews of Kaluza-Klein supergravity may be found in [15, 28].

5 Supermembranes with fewer supersymmetries

Interest in $AdS_4 \times X^7$ solutions of $D = 11$ supergravity waned for a while but was revived by the arrival of the $D = 11$ supermembrane [29]. In 1991 this 2-brane was recovered as a solution of the $D = 11$ supergravity theory preserving one half of the supersymmetry [30, 31]. Specifically, in the case that N branes with the same charge are stacked together, the metric is given by

$$ds^2 = (1 + Na^6/y^6)^{-2/3} dx^\mu dx_\mu + (1 + Na^6/y^6)^{1/3} (dy^2 + y^2 d\Omega_7^2) \quad (18)$$

and the four-form field strengths by

$$\tilde{F}_7 \equiv *F_4 = \pm 6Na^6 \epsilon_7 \quad (19)$$

Here $d\Omega_7^2$ corresponds to the round S^7 .

Of particular interest is the near horizon limit $y \rightarrow 0$, or equivalently the large N limit, because then the metric reduces to [32, 33, 34] the $AdS_4 \times S^7$ vacuum with

$$m^{-6} = Na^6 \quad (20)$$

Thus

$$ds^2 = y^4 m^4 dx^\mu dx_\mu + m^{-2} y^{-2} dy^2 + m^{-2} d\Omega_7^2 \quad (21)$$

which is just $AdS_4 \times S^7$ with the AdS metric written in horospherical coordinates.

Note that the round S^7 makes its appearance. The question naturally arises as to whether the compactifications with fewer supersymmetries discussed above also arise as near-horizon geometries of p -brane solitons. The answer is yes and the soliton solutions are easy to construct [35, 36]. One simply makes the replacement

$$d\Omega_7^2 \rightarrow d\hat{\Omega}_7^2 \quad (22)$$

in (18), where $d\hat{\Omega}_7^2$ is the metric on an arbitrary Einstein space X^7 with the same scalar curvature as the round S^7 . The space need only be Einstein; it need not be homogeneous [35]. (There also exist brane solutions on Ricci flat X^7 [35]). Note, however, that these non-round-spherical solutions do not tend to Minkowski space as $r \rightarrow \infty$. Instead the metric on the 8-dimensional space transverse to the brane is asymptotic to a generalized cone

$$ds_8^2 = dy^2 + y^2 d\hat{\Omega}_7^2 \quad (23)$$

and 8-dimensional translational invariance is absent except when X^7 is the round S^7 . Note, however that the solutions have no conical singularity at $y = 0$ since the metric tends to $AdS_4 \times X^7$. By introducing the Schwarzschild-like coordinate r given by

$$r^6 = y^6 + Na^6 \quad (24)$$

\mathcal{H}	<i>Space</i>	<i>restrictions</i>
$SO(n)$	R^n	$n \geq 2$
$U(m)$	$R^n \simeq C^m$	$m \geq 2, n = 2m$
$SU(m)$	$R^n \simeq C^m$	$m \geq 2, n = 2m$
$Sp(1) \times Sp(m)$	$R^n \simeq H^m$	$m \geq 1, n = 4m$
$Sp(m)$	$R^n \simeq H^m$	$m \geq 1, n = 4m$
G_2	R^7	
$Spin(7)$	R^8	
$Spin(9)$	R^{16}	

Table 2: Holonomies of Riemannian connections

we can see that the solutions exhibits an event-horizon at $r = N^{1/6}a$. Indeed the solution may be analytically continued down to $r = 0$ where there is a curvature singularity, albeit hidden by the event horizon [33].

As a matter of fact, there is a one-to-one correspondence between Killing spinors on Einstein manifolds X^7 satisfying $\tilde{D}_m \eta = 0$, $m = 1, 2 \dots 7$, and Killing spinors on the Ricci-flat cone (23) satisfying $D_M \eta = 0$, $M = 1, 2 \dots 8$, [37, 20]. So $N = 1$ can then be understood as G_2 holonomy on X^7 or $Spin(7)$ holonomy on the cone. Similarly, the weak holonomy 3-form (15) lifts to a covariantly constant self-dual 4-form on the cone.

6 A speculation on Spin(9)

In Berger's classification [38] of holonomy groups of Levi-Civita connections given in Table 2, there are three exceptional cases G_2 , $Spin(7)$ and $Spin(9)$. G_2 and $Spin(7)$ have already made their appearance in our story, but $Spin(9)$ seems an unlikely candidate since it corresponds to the holonomy of a *sixteen*-dimensional space (the Cayley plane), and that seems too high for $D = 11$ supergravity. Nevertheless, as we shall now describe, there is a way in which subgroups of $SO(16)$ can appear as holonomies in the theory. (The Cayley Plane, which is the 16-dimensional coset $F_4/Spin(9)$, has also featured in recent, but apparently unrelated, speculations on hidden mathematical structures in $D = 11$ [39]. Recent discussions on the mathematics of $Spin(9)$ may be found in [40, 41].)

The supermembranes discussed in the previous section preserve a fraction $\nu = N/16$ of the spacetime supersymmetry where $1 \leq N \leq 8$ is the number of Killing spinors on X^7 . (In the near-horizon limit this doubles.) Following [30], we can attempt to quantify this in terms of a holonomy even more generalized than that discussed in section (4), namely

$$\mathcal{H}_{gen} \subset SO(16) \tag{25}$$

where N is then given by the number of singlets appearing in the decomposition of the 16 of $SO(16)$ under \mathcal{H}_{gen} . In the case of the brane with a round S^7 , we

have $\mathcal{H}_{gen} = SO(8)$, the 16 decomposes into an 8 plus 8 singlets and $\nu = 1/2$. Whereas in the case of the brane with a squashed S^7 , we would have only one singlet and $\nu = 1/16$. The origin of this $SO(16)$ was speculated in [42] and proved in [43]. After making a $D = 11$ Lorentz-non-covariant three/eight split of the $D = 11$ supergravity field equations, the tangent space group $SO(1, 2) \times SO(8)$ gets enlarged to an $SO(1, 2) \times SO(16)$ under which the supersymmetry parameter transforms as a $(2, 16)$. Note that the 16 is the *vector* representation, even though ϵ is a spacetime spinor^d. This can be understood by noting, as we have already done, that the supercovariant derivative (2) involves not merely the Levi-Civita connection but extra terms involving the 4-form field strength F_{MNPQ} , a fact that tends to be underemphasized in recent works on holonomy and $D = 11$ supergravity (a notable exception being [44]).

Although we do not know of any examples, it is thus possible that some $D = 11$ field configuration could have

$$Spin(9) = \mathcal{H}_{gen} \subset SO(16) \quad (26)$$

However, the embedding is such that $16 \rightarrow 16$ and so such a configuration can preserve no supersymmetries, which may seem something of an anticlimax.

All this raises an interesting question: What is the number of possible supersymmetries allowed in M-theory? A priori, from the M-theory algebra [45], 32ν can be any integer from 0 to 32 [46]. Although there are supersymmetric backgrounds realizing the maximum 32, it was thought for a time that partially supersymmetric backgrounds were restricted to $0 \leq \nu \leq 1/2$, such as the above mentioned membranes. However, recent work on pp waves [47, 48, 49, 50, 51] and Gödel universes [52] has revealed certain values of 32ν lying between 16 and 32. Riemannian holonomy cannot account for these exotic fractions, so let us therefore take the generalized holonomy approach, discussed above. Of course, one is not obliged to make a three/eight split of M-theory, one could make any $d/(11 - d)$ split. For example, after a four/seven split, the tangent space group $SO(1, 3) \times SO(7)$ is enlarged to an $SO(1, 3) \times SU(8)$ [42, 53]. Similar remarks apply to Type IIB. It is curious to note that if the 32 supercharges always belong to representations such as $(2, 16)$, $(4, 8)$ etc, then 32ν , the number of singlets appearing in the decomposition, is restricted to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 20, 22, 24, 26, 28, 32. This is consistent with the data^e. However, a better understanding of the two/nine and one/ten splits is necessary before ruling out other values.

^dThis phenomenon had been noted long ago by Cremmer and Julia [16] when *dimensionally reducing* $D = 11$ supergravity (or Type IIB supergravity) to $D = 3$, but here we are claiming $SO(16)$ already in $D = 11$ (or $D = 10$).

^eNote added: Interestingly enough, after the completion of this work, a Gödel universe with 14 supersymmetries was discovered [124] which completes this list.

7 Hopf dualities

In recent times, both perturbative and non-perturbative effects of ten-dimensional superstring theory have been subsumed by an eleven-dimensional M theory whose low-energy limit is $D = 11$ supergravity [54]. In particular, we have the duality

$$M \text{ on } X \equiv IIA \text{ on } Y \quad (27)$$

In the first examples X was just a direct product $Y \times S^1$ and in this way one could give an $D = 11$ M -theory origin to $D = 10$ Type IIA objects by either wrapping around the S^1 or by reducing. For example, the IIA string comes from the $M2$ -brane by wrapping [55, 61, 57]; the $D6$ -brane in $D = 10$ may be interpreted as a Kaluza-Klein monopole in $D = 11$ [60, 61] and the $D2$ brane in $D = 10$ may be interpreted as an $M2$ -brane in $D = 11$ by dualizing a vector into a scalar on the 3-dimensional worldvolume [58, 59].

However, the duality (27) may be generalized to the so-called *Hopf duality* where X is a twisted product or $U(1)$ bundle over Y [63, 64, 65, 66, 67]. In [63], for example, such M -theory vacua with $N > 0$ supersymmetry were presented which, from the perspective of perturbative Type IIA string theory, have $N = 0$. They can emerge whenever the X^7 is a $U(1)$ bundle over a 6-manifold Y^6 . The missing superpartners are Dirichlet 0-branes. Someone unable to detect Ramond-Ramond charge would thus conclude that these worlds have no unbroken supersymmetry. In particular, the gravitinos (and also some of the gauge bosons) are 0-branes not seen in perturbation theory but which curiously remain massless however weak the string coupling. The simplest example of this phenomenon is provided by the maximally-symmetric S^7 . Considered as a compactification of $D = 11$ supergravity, the round S^7 yields a four dimensional AdS spacetime with $N = 8$ supersymmetry and $SO(8)$ gauge symmetry, for either orientation of S^7 . The Kaluza-Klein mass spectrum therefore falls into $SO(8)$ $N = 8$ supermultiplets. In particular, the massless sector is described [15] by gauged $N = 8$ supergravity [69]. Since S^7 is a $U(1)$ bundle over CP^3 the same field configuration is also a solution of $D = 10$ Type IIA supergravity [114]. However, the resulting vacuum has only $SU(4) \times U(1)$ symmetry and either $N = 6$ or $N = 0$ supersymmetry depending on the orientation of the S^7 . The reason for the discrepancy is that the modes charged under the $U(1)$ are associated with the Kaluza-Klein reduction from $D = 11$ to $D = 10$ and are hence absent from the Type IIA spectrum originating from the massless Type IIA supergravity. In other words, they are Dirichlet 0-branes [68] and hence absent from the perturbative string spectrum. There is thus more non-perturbative gauge symmetry and supersymmetry than perturbative. (Here the words “perturbative” and “non-perturbative” are shorthand for “with and without the inclusion of Dirichlet 0-branes”, but note that the Type IIA compactification has non-perturbative features even without the 0-branes [63]). The right-handed orientation is especially interesting because the perturbative theory has no supersymmetry at all! A summary of perturbative versus non-perturbative symmetries is given in Table 2. In particular, the non-perturbative vacuum may have unbroken supersymmetry even when the perturbative vacuum has none.

Compactification	Perturbative Type <i>IIA</i>		Nonperturbative M-theory	
Left round S^7	$N = 6$	$SU(4) \times U(1)$	$N = 8$	$SO(8)$
Right round S^7	$N = 0$	$SU(4) \times U(1)$	$N = 8$	$SO(8)$
Left squashed S^7	$N = 1$	$SO(5) \times U(1)$	$N = 1$	$SO(5) \times SU(2)$
Right squashed S^7	$N = 0$	$SO(5) \times U(1)$	$N = 0$	$SO(5) \times SU(2)$
Left $M(3, 2)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$	$N = 2$	$SU(3) \times SU(2) \times U(1)$
Right $M(3, 2)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$

Table 2: Perturbative versus non-perturbative symmetries.

We have focussed in this paper on compactifications from $D = 11$ but much of the discussion applies, *mutatis mutandis* to Type *IIB*. For example we have the Hopf T-duality

$$IIB \text{ on } S^5 \equiv IIA \text{ on } CP^2 \times S^1 \quad (28)$$

which untwists the S^5 [64].

8 Recent developments

Following the M-theory revolution of 1995, it was noted [72, 73, 74, 76, 75, 81, 83] that non-chiral $N = 1$ heterotic string compactifications can be dual to $D = 11$ supergravity compactified on Ricci-flat seven-dimensional spaces of G_2 holonomy [85, 86, 87, 84, 77]. See also [78, 79] for $N = 1$ compactifications on 8-manifolds of $spin(7)$ holonomy.

In 2002, G_2 manifolds continue play an important role in $D = 11$ M-theory for the same $N = 1$ reason as in 1982 . Of course, for phenomenology we require not only $N = 1$ but also chirality, and the lack of chirality on smooth seven-manifolds [80] was one of the main reasons that $D = 11$ supergravity fell out of favor. But the full M-theory, as opposed to its low energy limit, admits the possibility of *singular* G_2 compactifications which can yield *chiral* ($N = 1, D = 4$) models living in Minkowski space, with realistic gauge groups [88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111] and doublet-triplet splitting [112].

9 K3

In 1983 a second example of the utility of holonomy was provided by the compactification of $D = 11$ supergravity on the $K3$ [7]. This manifold had recently made its appearance in the physics literature as a gravitational instanton [3]. It is a self-dual, and hence Ricci flat, solution with $m = 0$ and $\mathcal{H} = SU(2)$ instead of the $SU(2) \times SU(2)$ of a generic 4-manifold. Hence $N = N_{max}/2$. $K3$ provided a novel phenomenon in Kaluza-Klein theory: the appearance of massless particles as a consequence of the *topology*, as opposed to the *geometry* of the compactifying

manifold, determined by Betti numbers and index theorems [7]. A discussion of boson and fermion zero modes on $K3$ and their relation to axial and conformal anomalies [70], may be found in [3, 71, 1].

$K3$ was thus the forerunner of the very influential *Calabi-Yau* compactifications^f of ten-dimensional supergravity and string theory [8] and, indeed, the Ricci-flat G_2 compactifications of M -theory mentioned above, some of which correspond to $K3$ fibrations [108].

$K3$ compactification allows for the possibility of chirality in the lower dimensional spacetime and chiral compactification of the Type IIB supergravity was undertaken in [116] and the heterotic string theory in [117].

In 1986, it was pointed out [115] that $D = 11$ supergravity on $R^{10-n} \times K3 \times T^{n-3}$ [7] and the $D = 10$ heterotic string on $R^{10-n} \times T^n$ [82] not only have the same supersymmetry but also the same moduli spaces of vacua, namely

$$\mathcal{M} = \frac{SO(16+n, n)}{SO(16+n) \times SO(n)} \quad (29)$$

It took almost a decade for this “coincidence” to be explained but we now know that M -theory on $R^{10-n} \times K3 \times T^{n-3}$ is dual to the heterotic string on $R^{10-n} \times T^n$ [56, 57].

One way to understand this is to note that, when wrapped around $K3$ with its 19 self-dual and 3 anti-self-dual 2-forms, the $d = 6$ worldvolume fields of the $M5$ -brane (or $D5$ -brane) $(B^-_{\mu\nu}, \lambda^I, \phi^{[IJ]})$ reduce to the $d = 2$ worldsheet fields of the heterotic string in $D = 7$ (or $D = 6$) [120, 121, 122]. The 2-form yields $(19, 3)$ left and right moving bosons, the spinors yield $(0, 8)$ fermions and the scalars yield $(5, 5)$ which add up to the correct worldsheet degrees of freedom of the heterotic string. A consistency check is provided by the derivation of the Yang-Mills and Lorentz Chern-Simons corrections to the Bianchi identity of the heterotic string starting from the $M5$ -brane Bianchi identity [113].

Heterotic strings on $K3$ can also be self-dual [118, 119]. Moreover, Heterotic strings on Calabi-Yau manifolds that are a T^3 fibration over a base Q can be dual to M -theory on X^7 that is $K3$ fibered over Q and has G_2 holonomy [108], which brings us back to where we started.

10 Conclusions

I first wrote about supergravity in a popular article for *New Scientist* in 1977 [123], where I said “Supergravity is theoretically very compelling, but it has yet to prove its worth by experiment”; a remark still unfortunately true at *Supergravity@25*. Let

^fIndeed, the possibility of going from 10 dimensions to 4 on a Ricci-flat 6-manifold with $SU(3)$ holonomy so as to get $N = 1$ in $D = 4$ occurred to Pope, Nilsson and myself while writing up the $K3$ paper in 1983. Since we were at UT, Austin, at the time, we consulted one or two of the distinguished mathematicians in the Mathematics Department there, but were told they had never heard of such a thing! Consequently our paper states “We do not know any solutions with $\mathcal{H} = SU(3)$ ”.

us hope that by the *Supergravity@50* conference, or before, we can say something different.

It is a privilege to have been a member of the Supergravity community these 25 years and I would like to say “Thanks for the memories” to its discoverers.

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References

- [1] E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. **B186** (1981) 412-428.
- [2] M. A. Awada, M. J. Duff and C. N. Pope, *$N = 8$ supergravity breaks down to $N = 1$* , Phys. Rev. Lett. **50** (1983) 294.
- [3] S. W. Hawking and C. N. Pope, *Symmetry Breaking By Instantons In Supergravity*, Nucl. Phys. **B146**, 381 (1978).
- [4] L. Alvarez-Gaume and D. Z. Freedman, *Ricci Flat Kahler Manifolds And Supersymmetry*, Phys. Lett. **B94**, 171 (1980).
- [5] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Spontaneous supersymmetry breaking by the squashed seven-sphere*, Phys. Rev. Lett. **50** (1983) 2043.
- [6] M. J. Duff and C. N. Pope, *Kaluza-Klein supergravity and the seven sphere*, in Supersymmetry and Supergravity 82, proceedings of the Trieste conference.
- [7] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Compactification of $D = 11$ supergravity on $K3 \times T^3$* , Phys. Lett. **B129** (1983) 39.
- [8] P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. **B258** (1985) 46.
- [9] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in eleven-dimensions*, Phys. Lett. **B76** (1978) 409.
- [10] S. Ferrara, D. Z. Freedman and P. van Nieuwenhuizen, *Progress toward a theory of supergravity*, Phys. Rev. **D13** (1976) 3214.
- [11] S. Deser and B. Zumino, *Consistent supergravity*, Phys. Lett. **B62** (1976) 335.
- [12] M. J. Duff and P. van Nieuwenhuizen, *Quantum inequivalence of different field representations*, Phys. Lett. **B94** (1980) 179.
- [13] A. Aurilia, H. Nicolai and P. K. Townsend, *Hidden Constants: The Theta Parameter Of QCD And The Cosmological Constant Of $N=8$ Supergravity*, Nucl. Phys. **B176**, 509 (1980).
- [14] P. G. O. Freund and M. A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97** (1980) 233.
- [15] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. **130** (1986) 1.
- [16] E. Cremmer and B. Julia, *The $SO(8)$ supergravity*, Nucl. Phys. **B159** (1979) 141.
- [17] F. Englert, *Spontaneous compactification of eleven dimensional supergravity*, Phys. Lett. **B119** (1982) 339.
- [18] P. G. O. Freund, *Four-dimensional supersymmetries from eleven-dimensional supergravity*, Phys. Lett. **B130** (1983) 173

- [19] A. Gray, *Weak holonomy groups*, Math. Z. **123** (1971) 290.
- [20] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. Spence, *Branes At Conical Singularities And Holography*, Adv. Theor. Math. Phys. **2** (1999) 1249, hep-th/9808014.
- [21] L. Castellani, R. D'Auria, P. Fre and P. van Nieuwenhuizen, *Holonomy groups, sesquidual torsion fields and $SU(8)$ in $d = 11$ supergravity*, J. Math. Phys. **25** (1984) 3209.
- [22] E. Sezgin, *The spectrum of the eleven-dimensional supergravity compactified on the round seven sphere*, Trieste preprint IC/83/220.
- [23] D. Z. Freedman, G. W. Gibbons and P. C. West, *Ten into four won't go*, Phys. Lett. **B124** (1983) 491.
- [24] D. N. Page and C. N. Pope, *New squashed solutions of $d = 11$ supergravity*, Phys. Lett. **B147** (1985) 55.
- [25] P. van Nieuwenhuizen and N. Warner, *Integrability conditions for Killing spinors*, Commun. Math. Phys. **93** (1984) 277.
- [26] F. Bais, H. Nicolai and P. van Nieuwenhuizen, *Geometry of coset spaces and massless modes of the squashed seven sphere in supergravity*, Nucl. Phys. **B228** (1983) 333.
- [27] N. P. Warner, *Some new extrema of the scalar potential of gauged $N = 8$ supergravity* Phys. Lett. **B128** (1983) 169.
- [28] L. Castellani, R. D'Auria and P. Fre, *Supergravity and superstrings: a geometric perspective*, (in 3 volumes), World Scientific Press, 1991.
- [29] E. Bergshoeff, E. Sezgin and P. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **B189** (1987) 75.
- [30] M. J. Duff and K. Stelle, *Multimembrane solutions of $d = 11$ supergravity*, Phys. Lett. **B253** (1991) 113.
- [31] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rep. **259** (1995) 213.
- [32] G. W. Gibbons and P. K. Townsend, *Vacuum interpolation in supergravity via super p -branes*, Phys. Rev. Lett. **71** (1993) 5223.
- [33] M. J. Duff, G. W. Gibbons and P. K. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. B. **332** (1994) 321.
- [34] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, *Higher-dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. **12** (1995) 297.
- [35] M.J. Duff, H. Lü, C.N. Pope and E. Sezgin, *Supermembranes with fewer supersymmetries*, Phys. Lett. **B371** (1996) 206, hep-th/9511162.
- [36] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, *G/H M-branes and AdS_{p+2} geometries*, hep-th/9803039.

- [37] C. Bar, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993) 509.
- [38] M. Berger, *Sur les groupes d'holonomie homogene des varietes a connexion affine et des varietes riemanniennes*, Bull. Soc. Math. France **83** (1955) 225
- [39] P. Ramond, *Algebraic dreams*, hep-th/0112261.
- [40] Th. Friedrich, *Weak Spin(9)-structures on 16-dimensional Riemannian manifolds*, Asian Journal in Mathematics, 5 (2001), 129 - 160.
- [41] Th. Friedrich, *Spin(9)-structures and connections with totally skew-symmetric torsion*, math.DG/0205149
- [42] M. J. Duff, *$E_8 \times SO(16)$ Symmetry of $d = 11$ Supergravity?*, *Quantum Field Theory and Quantum Statistics*, Essays in Honour of E. S. Fradkin (Eds. Batalin, Isham and Vilkovisky, Adam Hilger, 1986).
- [43] H. Nicolai, *$D = 11$ supergravity with local $SO(16)$ invariance*, Phys. Lett. **B187** (1987) 316.
- [44] T. Ali, *M-theory on seven-manifolds with G-fluxes*, hep-th/0111220
- [45] P. K. Townsend, *M-theory from its superalgebra*, hep-th/9712004.
- [46] J. P. Gauntlett and C. M. Hull, *BPS states with extra supersymmetry*, hep-th/9909098.
- [47] M. Cvetič, H. Lu and C. N. Pope, *Penrose limits, pp-waves and deformed conifolds*, hep-th/0203082.
- [48] J. Michelson, *(Twisted) toroidal compactification of pp-waves*, hep-th/0203140
- [49] M. Cvetič, H. Lu and C. N. Pope, *M-theory pp-waves, Penrose limits and supernumerary supersymmetries*, Nucl. Phys. B **644**, 65 (2002), hep-th/0203229.
- [50] J. P. Gauntlett and C. M. Hull, *pp-waves in 11-dimensions with extra supersymmetry*, hep-th/0203255.
- [51] J. Michelson, *A pp-wave with 26 supercharges*, hep-th/0206204
- [52] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, hep-th/0209114.
- [53] B. De Wit and H. Nicolai, *$d=11$ supergravity with local $SU(8)$ invariance*, Nucl. Phys. **B274** (1986) 363.
- [54] M. J. Duff, *The world in eleven dimensions: supergravity, supermembranes and M-theory*, I. O. P. Publishing (1999). <http://bookmark.iop.org/bookpge.htm?ID=628818102-24049-20169067-Y&book=815p>
- [55] M. J. Duff, P. Howe, T. Inami and K. S. Stelle, *Superstrings in $D = 10$ from supermembranes in $D = 11$* , Phys. Lett. **B191** (1987) 70.

- [56] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B438** (1995) 109.
- [57] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85.
- [58] M. J. Duff and J. X. Lu, *Type II p-branes: the brane scan revisited*, Nucl. Phys. **B390** (1993) 276.
- [59] P. Townsend, *D-Branes From M-Branes*, Phys. Lett. **B373** (1996) 68.
- [60] M. J. Duff and J. X. Lu, *Black and super p-branes in diverse dimensions*, Nucl. Phys. **B416** (1994) 301.
- [61] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. **B350** (1995) 184.
- [62] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *The Criterion for Vacuum Stability in Kaluza-Klein Supergravity*, Phys. Lett. **B139**, 154 (1984).
- [63] M. J. Duff, H. Lu and C. N. Pope, *Supersymmetry without supersymmetry*, Phys. Lett. **B409** (1997) 136, hep-th/9704186
- [64] M. J. Duff, H. Lü and C. N. Pope, *AdS₅ × S⁵ untwisted*, hep-th/9803061.
- [65] M. J. Duff, H. Lü and C. N. Pope, *AdS₃ × S³ (un)twisted and squashed*, hep-th/9807173.
- [66] E. Halyo, *Supergravity on AdS_{5/4} × Hopf fibrations and conformal field theories*, hep-th/9803193.
- [67] R. Minasian and D. Tsimpis, *Hopf reductions, fluxes and branes*, hep-th/0106266.
- [68] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724
- [69] B. De Wit and H. Nicolai, *N = 8 supergravity*, Nucl. Phys. **B208** (1982) 323.
- [70] S. M. Christensen and M. J. Duff, *Axial and Conformal Anomalies for Arbitrary Spin in Gravity and Supergravity*, Phys. Lett. **76B**, 571 (1978).
- [71] S. M. Christensen and M. J. Duff, *New Gravitational Index Theorems and Supertheorems*, Nucl. Phys. **B154**, 301 (1979).
- [72] A. C. Cadavid, A. Ceresole, R. D'Auria and S. Ferrara, *11-dimensional supergravity compactified on Calabi-Yau threefolds*, hep-th/9506144.
- [73] G. Papadopoulos and P. K. Townsend, *Compactification of D=11 Supergravity on spaces with exceptional holonomy*, Phys. Lett. **B357** (1995) 300.
- [74] J. H. Schwarz and A. Sen, *Type IIA dual of six-dimensional CHL compactifications*, Nucl. Phys. **B454** (1995) 427.
- [75] S. Chaudhuri and D. A. Lowe, *Type IIA-Heterotic duals with maximal supersymmetry*, Nucl. Phys. **B459** (1996) 113.

- [76] J. A. Harvey, D. A. Lowe and A. Strominger, *N = 1 string duality*, Phys. Lett. **B362** (1995) 65.
- [77] R. Bryant and S. Salamon, *On The Construction Of Some Complete Metrics With Exceptional Holonomy*, Duke Math. J. **58** (1989) 829.
- [78] K. Becker and M. Becker, *M-theory on eight-manifolds*, hep-th/9605053.
- [79] S. W. Hawking and M. M. Taylor-Robinson, *Bulk charges in eleven dimensions*, hep-th/9711042
- [80] E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, in Quantum Field Theory and the Fundamental Problems, Shelter Island 1983, proceedings.
- [81] B. S. Acharya, *N = 1 Heterotic-supergravity duality and Joyce manifolds*, hep-th/9508046
- [82] K. S. Narain, *New heterotic string theories in uncompactified dimensions < 10*, Phys. Lett. **B169** (1986) 41.
- [83] P. S. Aspinwall, *Some relationships between dualities in string theory*, hep-th/9508154.
- [84] G. W. Gibbons, D. N. Page, and C. N. Pope, *Einstein Metrics On S^3 , R^3 , and R^4 Bundles*, Commun. Math. Phys. **127** (1990) 529.
- [85] D. Joyce, *Compact Manifolds With Special Holonomy*, (Oxford University Press, 2000).
- [86] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 . Part I*, J. Diff. Geom. **43** (1996) 291.
- [87] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 . Part II*, J. Diff. Geom. **43** (1996) 329.
- [88] B. Acharya, *On Realising $\mathcal{N} = 1$ Super Yang-Mills In M-Theory*, hep-th/0011089.
- [89] M. F. Atiyah, J. Maldacena, and C. Vafa, *An M-Theory Flop As A Large N Duality*, hep-th/0011256.
- [90] M. F. Atiyah and E. Witten, *M-Theory Dynamics On A Manifold Of G_2 Holonomy*, hep-th/0107177.
- [91] M. Cvetič, G. Shiu, and A. M. Uranga, *Three Family Supersymmetric Standard-Like Models From Intersecting Brane Worlds*, hep-th/0107143.
- [92] E. Witten, *Anomaly Cancellation on Manifolds of G_2 -holonomy*, hep-th/0108165.
- [93] B. Acharya, *Confining Strings From G_2 Holonomy Spacetimes*, hep-th/0101206.
- [94] M. Aganagic and C. Vafa, *Mirror Symmetry And A G_2 Flop*, hep-th/0105225.
- [95] B. Acharya, *M-Theory, Joyce Orbifolds, And Super Yang-Mills*, Adv. Theor. Math. Phys. **3** (1999) 227, hep-th/9812205.

- [96] J. Gomis, *D-Branes, Holonomy and M-Theory*, hep-th/010311.
- [97] H. Partouche and B. Pioline, *Rolling Among G_2 Vacua*, JHEP **0103** (2001) 005, hep-th/0011130.
- [98] J. D. Edelstein and C. Nunez, *D6 branes and M-theory geometrical transitions from gauged supergravity*, JHEP **0104**, 028 (2001), hep-th/0103167
- [99] S. Kachru and J. McGreevy, *M-Theory On Manifolds of G_2 Holonomy and Type IIA Orientifolds*, hep-th/0103223.
- [100] P. Kaste, A. Kehagias, and H. Partouche, *Phases of Supersymmetric Gauge Theories From M-Theory On G_2 Manifolds*, hep-th/0104124.
- [101] Y. Konishi and M. Naka, *Coset Construction Of $Spin(7)$, G_2 Gravitational Instantons*, hep-th/0104208.
- [102] A. Brandhuber, J. Gomis, S. S. Gubser, and S. Gukov, *Gauge Theory At Large N And New G_2 Holonomy Metrics*, hep-th/0106034.
- [103] A. Brandhuber, *$G(2)$ holonomy spaces from invariant three-forms*, hep-th/0112113.
- [104] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, *Supersymmetric M3-Branes And G_2 Manifolds*, hep-th/0106026.
- [105] M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, *Resolved Branes And M-Theory On Special Holonomy Spaces*, hep-th/0106177.
- [106] M. Cvetič, G. W. Gibbons, J. T. Liu, H. Lu, and C. N. Pope, *A New Fractional D2-Brane, G_2 Holonomy, and T-Duality*, hep-th/0106162.
- [107] G. W. Gibbons, H. Lu, C. N. Pope and K. S. Stelle, *Supersymmetric domain walls from metrics of special holonomy*, hep-th/0108191.
- [108] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152
- [109] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, *A G_2 Unification of the Deformed and Resolved Conifolds*, hep-th/0112138.
- [110] J. A. Harvey and G. Moore, *Superpotentials And Membrane Instantons*, hep-th/9907026.
- [111] A. Bilal, J-P. Derendinger and K. Sfetsos, *(Weak) G_2 holonomy from self-duality flux and supersymmetry*, hep-th/0111274.
- [112] E. Witten, *Deconstruction, G_2 holonomy, and doublet-triplet splitting*, hep-ph/0201086.
- [113] M. J. Duff, J. T. Liu and R. Minasian, *Eleven Dimensional Origin of String/String Duality: A One Loop Test*, Nucl. Phys. **B452** (1995) 261, hep-th/9506126.
- [114] B.E.W. Nilsson and C.N. Pope, *Hopf fibration of $D = 11$ supergravity*, Class. Quantum Grav. **1** (1984) 499.

- [115] M. J. Duff and B. E. W. Nilsson, *Four-dimensional string theory from the K3 lattice*, Phys. Lett. **B175** (1986) 417.
- [116] P. K. Townsend, *A new anomaly free chiral supergravity theory from compactification on K3*, Phys. Lett. **B139** (1984) 283.
- [117] M. B. Green, J. H. Schwarz and P. C. West, *Anomaly-free chiral theories in six dimensions*, Nucl. Phys. **B254** (1985) 327.
- [118] M. J. Duff and R. Minasian, *Putting string / string duality to the test*, Nucl. Phys. **B436**, 507 (1995), hep-th/9406198.
- [119] M. J. Duff, R. Minasian and E. Witten, *Evidence for Heterotic/Heterotic Duality*, Nucl. Phys. **B465**, 413 (1996), hep-th/9601036.
- [120] P. K. Townsend, *String - membrane duality in seven-dimensions*, Phys. Lett. **B354**, 247 (1995), hep-th/9504095.
- [121] J. A. Harvey and A. Strominger, *The heterotic string is a soliton*, Nucl. Phys. **B449**, 535 (1995) [Erratum-ibid. **B458**, 456 (1995)] hep-th/9504047.
- [122] S. Cherkis and J. H. Schwarz, *Wrapping the M theory fivebrane on K3*, hep-th/9703062.
- [123] M. J. Duff, *What's up with Gravity?*, New Scientist, January 13, 1977.
- [124] T. Harmark and T. Takayanagi, *Supersymmetric Gödel universes in string theory*, hep-th/0301206.